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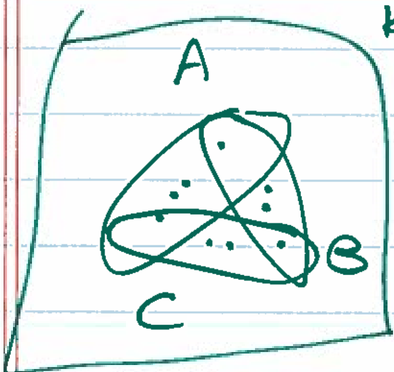
Intersecting Families of Permutations

What is an int. family?

classical example $\mathcal{H}$

k-uniform hypergraph on $[n]$

$\binom{[n]}{k}$ all possible edges

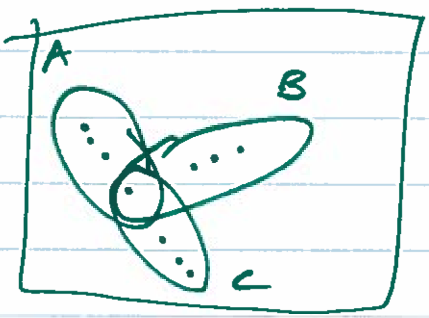


$\mathcal{H}$ is intersecting if $\forall A, B \in E(\mathcal{H})$

$|A \cap B| \geq 1$

(share at least one vertex)

4-uniform



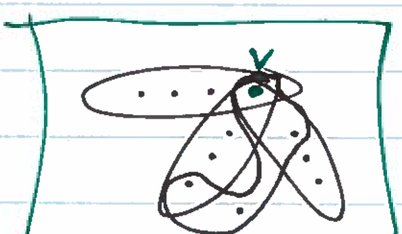
also intersecting

trivial: all edges share at least one vertex

A natural question in extremal combinatorics:
How large can an int. family be?

a candidate

fix vertex v put in all possible k-element ~~subsets~~ edges on $[n]$ containing v



int: (all edges share v)

large: $\binom{n-1}{k-1}$ edges

Thm [Erdős - Ko - Rado 1961]

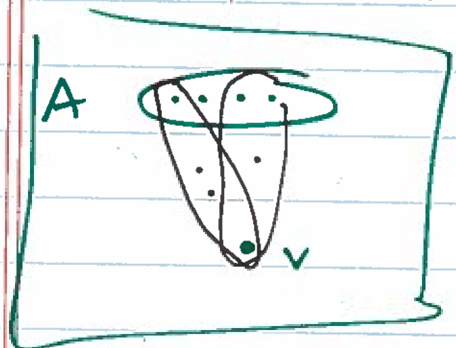
If $n \geq 2k$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting then $e(\mathcal{F}) \leq \binom{n-1}{k-1}$.

Furthermore, for $n > 2k$ we have equality only if $\mathcal{F}$ is trivial.

How large can a non-trivial int family be?

Thm [Hilton – Milner 1967]
for $n > 2k$, the largest non trivial int $\mathcal{F} \subseteq \binom{[n]}{k}$ have size

$$\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$



fix k-set A
fix $v \notin A$
put in all edges including v and int A

$$\binom{n-1}{k-1} + 1 - \binom{n-k-1}{k-1}$$

max. trivial int v A subtract edges not through A

Similar concepts for other discrete structures.
What about permutations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$\sigma(1) = 1$ $\sigma(2) = 3$ $\sigma(3) = 4$ $\sigma(4) = 2$ identity

1 3 4 2   1 2 3 4

~~1 2 4~~
~~1~~
~~4 2~~

1	2	3	4
1	3	4	2
2	3	1	4

- $\mathcal{F} \subseteq S_n$ is int. if $\forall \sigma, \tau \in \mathcal{F}$
$|\sigma \cap \tau| := |\{i \in [n] : \sigma(i) = \tau(i)\}| \geq 1$

1 2 3 4
1 3 4 2   trivial $\exists\, i, j$ s.t. $\sigma(i) = j \ \forall \sigma \in \mathcal{F}$
1 4 3 2

EKR type result

Thm [Frankl Deza 1977]

If $\mathcal{F} \subseteq S_n$ is int. then $|\mathcal{F}| \leq (n-1)!$

Thm [Cameron-Ku 2003, Larose-Malvenuto 2003]

We have equality only if $\mathcal{F} \subseteq S_n$ is trivial

HM type result

Thm [Ellis 2008]

For n suff. large, the largest non-trivial int $\mathcal{F} \subseteq S_n$ has size

$$\left(1 - \frac{1}{e} + o(1)\right)(n-1)!$$

$$\boxed{\begin{array}{cccc} 2 & 1 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \\ 1 & 3 & 2 & 4 \end{array}}$$

take (12)

and $\{\sigma \in S_n : \sigma(1) = 1,\ \sigma(j) = j \text{ for some } j \geq 2\}$

MAIN RESULT

Thm [BDDLS 2015]

The number of int. families of permutations is

$$\left(n^2 + o(1)\right) 2^{(n-1)!}.$$

Almost every int. family of permutations is trivial.

proof sketch. general framework

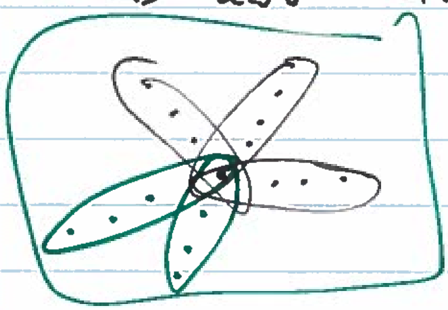
N_0 size of largest trivial int. family

N_1 size of largest non-trivial int family

M upper bound on # of maximal int. families

Obs 1

Any subset of a trivial int. family is also trivial.

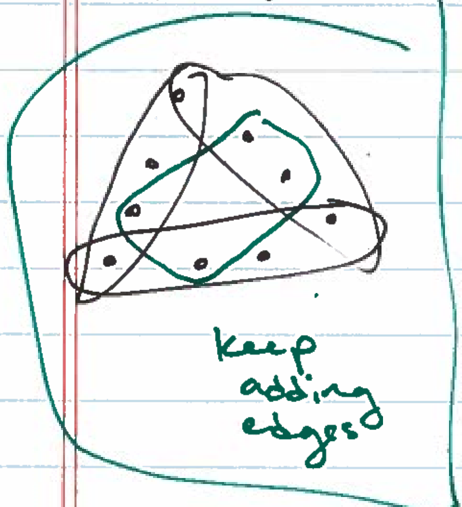


There are at least 2^{N_0} trivial int. families

Obs 2

Any nontrivial int family is contained in a maximal non-trivial int family.

maximal cannot be extended to a larger int family



There are at most $M \cdot 2^{N_1}$ nontrivial int. families

If $\dfrac{M \cdot 2^{N_1}}{2^{N_0}} \to 0$

then say trivial int families are "typical". $\square$

Need a good enough bound on M.

Prop [BDDLS 2015]

The number of maximal int $\mathcal{F} \subseteq S_n$ is at most $\displaystyle\sum_{i=0}^{\frac{1}{2}\binom{2n}{n}} \binom{n!}{i} < n^{n2^{2n-1}}$.

proof.

for any $\mathcal{G} \subseteq S_n$

$$\mathcal{I}(\mathcal{G}) := \{\pi \in S_n : \forall \sigma \in \mathcal{G} \quad |\pi \wedge \sigma| \geq 1\}$$

$\mathcal{H}$ int iff $\mathcal{H} \subseteq \mathcal{I}(\mathcal{H})$

$\mathcal{H}$ max int iff $\mathcal{H} = \mathcal{I}(\mathcal{H})$

Let $\mathcal{F} \subseteq S_n$ be maximal int

we say $\mathcal{G} \subseteq \mathcal{F}$ is a generating set if $\mathcal{F} = \mathcal{I}(\mathcal{G})$

pick a minimal gen set of $\mathcal{F}$, $\mathcal{F}_0 = \{\sigma_1, \ldots, \sigma_s\}$.

consider $\mathcal{F}_0 \setminus \{\sigma_j\}$

$$\mathcal{F} \subseteq \mathcal{I}(\mathcal{F}_0 \setminus \{\sigma_j\})$$

but $\mathcal{F} \neq \mathcal{I}(\mathcal{F}_0 \setminus \{\sigma_j\})$ by minimality of $\mathcal{F}_0$

so $\forall j \in [s]$ $\exists \tau_j \in \mathcal{I}(\mathcal{F}_0 \setminus \{\sigma_j\}) \setminus \mathcal{F}$.

$\forall i \neq j$, $|\tau_i \wedge \sigma_j| \geq 1$ $\qquad \tau_j \in \mathcal{I}(\mathcal{F}_0 \setminus \{\sigma_j\})$ $\ni \sigma_i$

$|\tau_j \wedge \sigma_j| < 1$ $\qquad \tau_j \notin \mathcal{F} = \mathcal{I}(\mathcal{F}_0)$

Thm [Bollobás 1965].

Let $A_1, \ldots, A_s$ be sets of size a and $B_1, \ldots, B_s$ be sets of size b st

$|A_i \cap B_i| < 1$ and $|A_i \cap B_j| \geq 1$

for every $1 \leq i, j \leq s$. Then

$$s \leq \binom{a+b}{a}.$$

for any $\pi \in S_n$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$H_\pi := \{(1, \pi(1)), \dots, (n, \pi(n))\}$$

n element set of ordered pairs

$$\pi \quad 1\ 3\ 4\ 2$$

$$H_\pi \quad \{(1,1), (2,3), (3,4), (4,2)\}$$

so $\mathcal{F} \subseteq S_n$ becomes n-uniform hypergraph on $[n]\times[n]$

for any $\pi, \pi' \in S_n$

$$|H_\pi \cap H_{\pi'}| = |\pi \cap \pi'|$$

Let $A_i = H_{\sigma_i}$ and $B_i = H_{\tau_i}$

$$s = |\mathcal{F}_0| \leq \binom{2n}{n}$$

count gen sets.

$\mathcal{F}_0$ not nec unique gen set of $\mathcal{F}$

$$\mathcal{F} = I(\mathcal{F}_0)$$

$\mathcal{F} \mapsto \mathcal{F}_0$ injection

maximal int families is bounded by the number of sets of at most $\binom{2n}{n}$ permutations

$\mathcal{F} \subseteq S_n$ is <u>t-int</u> if $\forall \sigma, \pi \in \mathcal{F}$ $|\sigma \cap \pi| \geq t$.

Thm [BDDLS 2015]

The number of t-int families of S_n is

$$\left(\binom{n}{t}^2 t! + o(1)\right) 2^{(n-t)!}$$

Almost every t-int family is trivial.

Thm [BDDLS 2015]
The number of t-int families of $\binom{[n]}{k}$ is

$$\left(\binom{n}{t} + o(1)\right) 2^{\binom{n-t}{k-t}}$$

Almost every t-int family is trivial.

Thm [BDDLS 2015] if $k \geq 2$ and either $q=2,\ n \geq 2k+2$ or $q \geq 3,\ n \geq 2k+1$
The number of int families of k-dim. subspaces of $\mathbb{F}_q^n$ is

$$\left(\begin{bmatrix} n \\ 1 \end{bmatrix}_q + o(1)\right) 2^{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q}.$$

Almost every int family is trivial

Gaussian binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \prod_{i=0}^{k-1} \frac{q^{n-i}-1}{q^{k-i}-1}.$$

\# k-dim subspaces in n-dim vectorspace V

a family $\mathcal{F}$ of k-dim subspaces of V is intersecting if $\dim(F_1 \cap F_2) \geq 1$ $\forall F_1, F_2 \in \mathcal{F}$.

Hsieh 1975 EKR $n \geq 2k+1$ $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$